The hook fusion procedure for Hecke algebras

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Abstract

We derive a new expression for the q-analogue of the Young symmetrizer which generate irreducible representations of the Hecke algebra. We obtain this new expression using Cherednik's fusion procedure. However, instead of splitting Young diagrams into their rows or columns, we consider their principal hooks. This minimises the number of auxiliary parameters needed in the fusion procedure.

1 Introduction

In this article we present a new expression for certain elements in the Hecke algbra, related to the q-analogue of the Young symmetrizer, which generate its irreducible representations. We will obtain this new expression using Cherednik's fusion procedure. This method originates from the work of Jucys [9], and has already been used by Nazarov and Tarasov [12, 13, 14]. However our approach differs by minimising the number of auxiliary parameters needed in the fusion procedure. This is done by considering hooks of Young diagrams, rather than their rows or columns as in [12, 13, 14].

Let H_n be the finite dimensional Hecke algebra over the field $\mathbb{C}(q)$ of rational functions in q, with the generators $T_1, \ldots, T_{n-1}$ and the relations

$$(T_i - q)(T_i + q^{-1}) = 0; \tag{1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}; \tag{2}$$

$$T_i T_j = T_j T_i, \quad j \neq i, i+1 \tag{3}$$

for all possible indices i and j.

The generators $T_1, \ldots, T_{n-1}$ are invertible since

$$T_i^{-1} = T_i - q + q^{-1} \tag{4}$$

due to (1).

For any index $i = 1, \ldots, n-1$ let $\sigma_i = (i, i+1)$ be the adjacent transposition in the symmetric group S_n. Take any element $\sigma \in S_n$ and choose a reduced decomposition $\sigma = \sigma_{i_1} \ldots \sigma_{i_l}$. As

usual put $T_\sigma = T_{i_1} \ldots T_{i_l}$, this element of the algebra H_n does not depend on the choice of reduced decomposition of σ due to (2) and (3). The element of maximal length in S_n will be denoted by σ_0. We will write T_0 instead of T_{σ_0} for short. The elements T_σ form a basis of H_n as a vector space over the field $\mathbb{C}(q)$. We will also use the basis in H_n formed by the elements T_σ^{-1}.

A *partition* of n is a sequence of weakly decreasing integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$ whose sum is equal to n. The *Young diagram* of a partition λ is the set of boxes $(i, j) \in \mathbb{Z}^2$ such that $1 \leqslant j \leqslant \lambda_i$. In drawing such diagrams we let the first coordinate i increase as one goes downwards, and the second coordinate j increase from left to right. For example the partition $\lambda = (3, 3, 2)$ gives the diagram
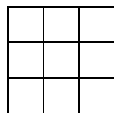
If (i, j) is a box in the diagram of λ, then the (i, j)-*hook* is the set of boxes in λ

$$\{(i, j') : j' \geqslant j\} \cup \{(i', j) : i' \geqslant i\},$$

We call the (i, i)-hook the ith *principal hook*.

A *standard tableau*, Λ, is a filling of the diagram λ in which the entries are the numbers 1 to n, each occurring once. If the box (i, j) contains a we define the *content* of the box to be $c_a(\Lambda) = j - i$.

The $\mathbb{C}(q)$-algebra H_n is semisimple; see [8, Section 4] for a short proof of this well known fact. The simple ideals of H_n are labeled by partitions λ of n, like the equivalence classes of irreducible representations of the symmetric group S_n.

In this article, for any standard tableau Λ of shape λ we will construct a certain non-zero element $F_\Lambda \in H_n$. Under left multiplication by the elements of H_n, the left ideal $H_n F_\Lambda \subset H_n$ is an irreducible H_n-module. The H_n-modules V_λ for different partitions λ are pairwise non-equivalent; see Corollary 3.1. At $q = 1$, the algebra $H_n(q)$ specializes to the group ring $\mathbb{C}S_n$, where T_σ becomes the permutation $\sigma \in S_n$. The $H_n(q)$-module V_λ then specializes to the irreducible representation of S_n, coresponding to the partition λ, [16].

Our construction of V_λ employs a certain limiting process called the *fusion procedure*. The idea of this construction goes back to [2, Section 3] were no proofs were given however. The element F_Λ is related to the q-analogue of the Young symmetrizer in the group ring $\mathbb{C}S_n$ constructed in [7].

For each $i = 1, \ldots, n-1$ introduce the H_n-valued rational function in two variables $a \neq 0$, $a \neq b \in \mathbb{C}(q)$

$$F_i(a, b) = T_i + \frac{q - q^{-1}}{a^{-1}b - 1}. \tag{5}$$

Now introduce n variables $z_1, \ldots, z_n \in \mathbb{C}(q)$. Equip the set of all pairs (i, j) where $1 \leqslant i < j \leqslant n$, with the following ordering. The pair (i, j) precedes another pair (i', j') if $j < j'$, or if $j = j'$ but $i < i'$. Call this the *reverse-lexicographic* ordering. Take the ordered product

$$F_\Lambda(z_1, \ldots, z_n) = \prod_{(i,j)}^{\rightarrow} F_{j-i}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j) \tag{6}$$

over this set. Consider the product (6) as a rational function taking values in H_n, of the variables $z_1, \ldots, z_n$. If i and j sit on the same diagonal in the tableau Λ, then $F_{j-i}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j)$ has a pole at $z_i = z_j \neq 0$.

Let $\mathcal{R}_\Lambda$ be the vector subspace in $\mathbb{C}(q)^n$ consisting of all tuples $(z_1, \ldots, z_n)$ such that $z_i = z_j$ whenever the numbers i and j appear in the same row of the tableau Λ.

As a direct calculation using (1) and (2) shows, these functions satisfy

$$F_i(a, b) F_{i+1}(a, c) F_i(b, c) = F_{i+1}(b, c) F_i(a, c) F_{i+1}(a, b). \tag{7}$$

Due to (3) these rational functions also satisfy the relations

$$F_i(a, b) F_j(c, d) = F_j(c, d) F_i(a, b); \qquad j \neq i, i+1. \tag{8}$$

Using (7) and (8) we may reorder the product $F_\Lambda(z_1, \ldots, z_n)$ such that each singularity is contained in an expression known to be regular at $z_1 = z_2 = \ldots = z_n \neq 0$, [13]. It is by this method that it was shown that the restriction of the rational function $F_\Lambda(z_1, \ldots, z_n)$ to the subspace $\mathcal{R}_\Lambda$ is regular at $z_1 = z_2 = \ldots = z_n \neq 0$. Furthermore the following theorem was proved;

Theorem 1.1 *Restriction to $\mathcal{R}_\Lambda$ of the rational function $F_\Lambda(z_1, \ldots, z_n)$ is regular at $z_1 = z_2 = \ldots = z_n \neq 0$ and has value $F_\Lambda \in H_n$. The left ideal generated by this element is irreducible, and the H_n-modules for different partitions λ are pairwise non-equivalent.*

Similarly, we may form another expression for F_Λ by considering the subspace in $\mathbb{C}(q)^n$ consisting of all tuples $(z_1, \ldots, z_n)$ such that $z_i = z_j$ whenever the numbers i and j appear in the same column of the tableau Λ [12].

In this article we present a new expression for the element F_Λ which minimises the number of auxiliary parameters needed in the fusion procedure. We do this by considering hooks of standard tableaux rather than their rows or columns.

Let $\mathcal{H}_\Lambda$ be the vector subspace in $\mathbb{C}(q)^n$ consisting of all tuples $(z_1, \ldots, z_n)$ such that $z_i = z_j$ whenever the numbers i and j appear in the same principal hook of the tableau Λ. We will prove the following theorem.

Theorem 1.2 *Restriction to $\mathcal{H}_\Lambda$ of the rational function $F_\Lambda(z_1, \ldots, z_n)$ is regular at $z_1 = z_2 = \ldots = z_n \neq 0$ and has value $F_\Lambda \in H_n$. The left ideal generated by this element is irreducible, and the H_n-modules for different partitions λ are pairwise non-equivalent. The element is the same as the element in Theorem 1.1.*

In particular, this hook fusion procedure can be used to form irreducible representations of H_n corresponding to Young diagrams of hook shape using only one auxiliary parameter, z. By taking this parameter to be 1 we find that no parameters are needed for diagrams of hook shape. Therefore if ν is a partition of hook shape, and N a standard tableau of shape ν, we have

$$F_N = F_N(z) = \overrightarrow{\prod_{(p,q)}} F_{j-i}(q^{2c_i(\Lambda)}, q^{c_j(\Lambda)}), \tag{9}$$

with the pairs (i, j) in the product ordered reverse-lexicographically.

To motivate the study of modules corresponding to partitions of hook shape first let us consider the *Jacobi-Trudi identities* [10, Chapter I3]. There is an isomorphism from the ring of symmetric functions to the Grothendieck ring of representations of the Hecke algebra. Therefore we can think of the following identities as dual to the Jacobi-Trudi identities.

If $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $n = \lambda_1 + \ldots + \lambda_k$ then we have the following decomposition of the induced representation of the tensor product of modules corresponding to the rows of λ;

$$\mathrm{Ind}_{H_{\lambda_1} \times H_{\lambda_2} \times \cdots \times H_{\lambda_k}}^{H_n} V_{(\lambda_1)} \otimes V_{(\lambda_2)} \otimes \cdots \otimes V_{(\lambda_k)} \cong \bigoplus_\mu (V_\mu)^{\oplus K_{\mu\lambda}},$$

where the sum is over all partitions of n. Note that $V_{(\lambda_i)}$ is the trivial representation of H_{λ_i}, that sends generators T_i to q. The coefficients $K_{\mu\lambda}$ are non-negative integers known as *Kostka numbers*, [10]. Importantly, we have $K_{\lambda\lambda} = 1$.

On the subspace $\mathcal{R}_\Lambda$, if $z_i/z_j \notin q^{\mathbb{Z}}$ when i and j are in different rows of Λ then the above induced module may be realised as the left ideal in H_n generated by $F_\Lambda(z_1, \ldots, z_n)$. The irreducible representation V_λ appears in the decomposition of this induced module with

coefficient 1, and is the ideal of H_n generated by $F_\Lambda(z_1, \ldots, z_n)$ when $z_1 = z_2 = \ldots = z_n \neq 0$. The fusion procedure of Theorem 1.1 provides a way of singling out this irreducible component.

Similarly we have the equivalent identity for columns,

$$\mathrm{Ind}_{H_{\lambda_1'} \times H_{\lambda_2'} \times \cdots \times H_{\lambda_l'}}^{H_n} V_{(1^{\lambda_1'})} \otimes V_{(1^{\lambda_2'})} \otimes \cdots \otimes V_{(1^{\lambda_l'})} \cong \bigoplus_\mu (V_\mu)^{\oplus K_{\mu'\lambda'}},$$

where l is the number of columns of λ. In this case $V_{(1^{\lambda_i'})}$ is the alternating representation of $H_{\lambda_i'}$. This induced module is isomorphic to the left ideal of H_n generated by $F_\Lambda(z_1, \ldots, z_n)$ considered on the subspace $\mathcal{R}_{\Lambda'}$, with $z_i/z_j \notin q^{\mathbb{Z}}$ when i, j are in different columns of Λ. Again the irreducible representation V_λ appears in the decomposition of this induced module with coefficient 1, and is the ideal of H_n generated by $F_\Lambda(z_1, \ldots, z_n)$ when $z_1 = z_2 = \ldots = z_n \neq 0$.

There is another expression known as the *Giambelli identity* [4]. Unlike the Jacobi-Trudi identities, this identity involves splitting λ into its principal hooks, rather than its rows or columns. A combinatorial proof of the Giambelli identity can be found in [3].

Divide a Young diagram λ into boxes with positive and non-positive content. We may illustrate this on the Young diagram by drawing 'steps' above the main diagonal. Denote the boxes above the steps by $\alpha(\lambda)$ and the rest by $\beta(\lambda)$. For example, the following figure illustrates λ, $\alpha(\lambda)$ and $\beta(\lambda)$ for $\lambda = (3, 3, 2)$.
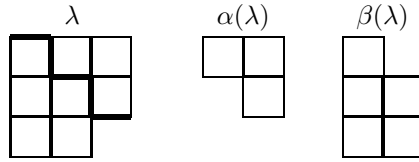

Figure 1: The Young diagram $(3, 3, 2)$ divided into boxes with positive content and non-positive content

If we denote the rows of $\alpha(\lambda)$ by $\alpha_1 > \alpha_2 > \ldots > \alpha_d > 0$ and the columns of $\beta(\lambda)$ by $\beta_1 > \beta_2 > \ldots > \beta_d > 0$, then we have the following alternative notation for λ;

$$\lambda = (\alpha|\beta),$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$.

Here d denotes the length of the side of the *Durfee square* of shape λ, which is the set of boxes corresponding to the largest square that fits inside λ, and is equal to the number of principal hooks in λ. In our example $d = 2$ and $\lambda = (2, 1|3, 2)$.

We may consider the following identity as a dual of the Giambelli identity.

$$\mathrm{Ind}_{H_{h_1} \times H_{h_2} \times \cdots \times H_{h_d}}^{H_n} V_{(\alpha_1|\beta_1)} \otimes V_{(\alpha_2|\beta_2)} \otimes \cdots \otimes V_{(\alpha_d|\beta_d)} \cong \bigoplus_{\mu} (V_\mu)^{\oplus D_{\mu\lambda}},$$

where h_i is the length of the ith principal hook, and the sum is over all partitions of n. This is a decomposition of the induced representation of the tensor product of modules of hook shape. Further these hooks are the principal hooks of λ. The coefficients, $D_{\mu\lambda}$, are non-negative integers, and in particular $D_{\lambda\lambda} = 1$.

On the subspace $\mathcal{H}_\Lambda$, if $z_i/z_j \notin q^{\mathbb{Z}}$ when i and j are in different principal hooks of Λ then the above induced module may be realised as the left ideal in H_n generated by $F_\Lambda(z_1, \ldots, z_n)$. The irreducible representation V_λ appears in the decomposition of this induced module with coefficient 1, and is the ideal of H_n generated by $F_\Lambda(z_1, \ldots, z_n)$ when $z_1 = z_2 = \ldots = z_n$.

Hence, in this way, our hook fusion procedure relates to the Giambelli identity in the same way that Cherednik's original fusion procedure relates to the Jacobi-Trudi identity. Namely, it provides a way of singling out the irreducible component V_λ from the above induced module.

The fusion procedure was originally developed in the study of affine Hecke algebras, [1]. Our results may be regarded as an application of the representation theory of these algebras, [15]. Descriptions of the fusion procedure for the Symmetric group may be found in [14] and [5]. The hook fusion procedure for the Symmetric group was considered in [6].

2   Fusion Procedure for a Young Diagram

We fill a diagram λ by hooks to form a tableau Λ° in the following way: For the first principal hook we fill the column with entries 1, 2, ..., r and then fill the row with entries $r+1$, $r+2$, ..., s. We then fill the column of the second principal hook with $s+1$, $s+2$, ..., t and fill the row with $t+1$, $t+2$, ..., x. Continuing in this way we form the hook tableau.

Example 2.1. On the left is the hook tableau of the diagram $\lambda = (3, 3, 2)$, and on the right the same diagram with the content of each box.

1	4	5
2	6	8
3	7	

0	1	2
-1	0	1
-2	-1	

Therefore the sequence $(c_1(\Lambda^\circ), c_2(\Lambda^\circ), \ldots, c_8(\Lambda^\circ))$ is given by $(0, -1, -2, 1, 2, 0, -1, 1)$. ∎

Consider (6) as a rational function of the variables $z_1, \ldots, z_n$ with values in H_n. Using the substitution

$$w_i = q^{c_i(\Lambda)} z_i, \tag{10}$$

the factor $F_i(w_a, w_b)$ has a pole at $z_a = z_b$ if and only if the numbers a and b stand on the same diagonal of a tableau Λ. We then call the pair (a, b) a *singularity*. And we call the corresponding term $F_i(w_a, w_b)$ a *singularity term*, or simply a singularity.

Let a and b be in the same principal hook of Λ. If a and b are next to one another in the column of the hook then, on $\mathcal{H}_\lambda$, $F_i(w_a, w_b) = T_i - q$. Since

$$(T_i - q)^2 = -(q + q^{-1})(T_i - q) \tag{11}$$

then $\frac{-1}{q+q^{-1}} F_i(w_a, w_b)$ is an idempotent. Denote this idempotent P_i^-.

Similarly, if a and b are next to one another in the same row of the hook then $F_i(w_a, w_b) = T_i + q^{-1}$. And since

$$(T_i + q^{-1})^2 = (q + q^{-1})(T_i + q^{-1}) \tag{12}$$

then $\frac{1}{q+q^{-1}} F_i(w_a, w_b)$ is an idempotent. Denote this idempotent P_i^+.

We also have

$$F_i(a, b) F_i(b, a) = 1 - \frac{(q - q^{-1})^2 ab}{(a - b)^2}. \tag{13}$$

Therefore, if the contents $c_a(\Lambda)$ and $c_b(\Lambda)$ differ by a number greater than one, then the factor $F_i(w_a, w_b)$ is invertible in H_n when $z_a = z_b \neq 0$ for all values of q.

The presence of singularity terms in the product $F_\Lambda(z_1, \ldots, z_n)$ mean this product may or may not be regular on the vector subspace of $\mathcal{H}_\lambda$ consisting of all tuples $(z_1, \ldots, z_n)$ such that $z_1 = z_2 = \ldots = z_n \neq 0$. Using the following lemma, we will be able to show that $F_\Lambda(z_1, \ldots, z_n)$ is indeed regular on this subspace.

Lemma 2.2 *Restriction of the rational function $F_i(a, b) F_{i+1}(a, c) F_i(b, c)$ to the set of (a, b, c) such that $a = q^{\pm 2} b$, is regular at $a = c \neq 0$.*

Proof. Let us expand the product at the left hand side of (7) in the factor $F_{i+1}(a, c)$. By the definition (5) we will get the sum

$$F_i(a, b)T_{i+1}F_i(b, c) + \frac{q - q^{-1}}{a^{-1}c - 1}F_i(a, b)F_i(b, c).$$

Here the restriction to $a = q^{\pm 2}b$ of the first summand is evidently regular at $a = c$. After the substitution $b = q^{\mp 2}a$, the second summand takes the form

$$\frac{q - q^{-1}}{a^{-1}c - 1}(T_i \mp q^{\pm 1})\left(T_i + \frac{q - q^{-1}}{q^{\pm 2}a^{-1}c - 1}\right) = \frac{q - q^{-1}}{a^{-1}c - q^{\mp 2}}(q^{\pm 1} \mp T_i).$$

The rational function of a, c at the right hand side of the last displayed equality is also evidently regular at $a = c$. ∎

In particular, if the middle term on the left hand side of (7) is a singularity and the other two terms are an appropriate idempotent and *triple term*, then the three term product, or *triple* is regular at $z_1 = z_2 = \ldots = z_n \neq 0$. we may now prove the first statement of Theorem 1.2.

Proposition 2.3 *The restriction of the rational function $F_\Lambda(z_1, \ldots, z_n)$ to the subspace $\mathcal{H}_\lambda$ is regular at $z_1 = z_2 = \ldots = z_n \neq 0$.*

Proof. Consider any standard tableau Λ' obtained from the tableau Λ by an adjacent transposition of its entries, say by $\sigma_k \in S_n$. Using the relations (7) and (8), we derive the equality of rational functions in the variables $z_1, \ldots, z_n$

$$F_\Lambda(z_1, \ldots, z_n)F_{n-k}(q^{2c_{k+1}(\Lambda)}z_{k+1}, q^{2c_k(\Lambda)}z_k) =$$

$$F_k(q^{2c_k(\Lambda)}z_k, q^{2c_{k+1}(\Lambda)}z_{k+1})F_{\Lambda'}(z_1', \ldots, z_n'), \tag{14}$$

where the sequence of variables $(z_1', \ldots, z_n')$ is obtained from the sequence $(z_1, \ldots, z_n)$ by exchanging the terms z_k and z_{k+1}. Observe that

$$(z_1', \ldots, z_n') \in \mathcal{H}_{\Lambda'} \quad \Leftrightarrow \quad (z_1, \ldots, z_n) \in \mathcal{H}_\Lambda.$$

Also observe that here $|c_k(\Lambda) - c_{k+1}(\Lambda)| \geqslant 2$ because the tableaux Λ and Λ' are standard. Therefore the functions

$$F_k(q^{2c_k(\Lambda)}z_k, q^{2c_{k+1}(\Lambda)}z_{k+1}) \quad \text{and} \quad F_{n-k}(q^{2c_{k+1}(\Lambda)}z_{k+1}, q^{2c_k(\Lambda)}z_k)$$

appearing in the equality (14), are regular at $z_k = z_{k+1} \neq 0$. Moreover, their values at $z_k = z_{k+1} \neq 0$ are invertible in the algebra H_n, see the relation (13). Due to these two

observations, the equality (14) shows that Proposition 2.3 is equivalent to its counterpart for the tableau Λ' instead of Λ.

Let us take the hook tableau Λ° of shape λ. There is a chain $\Lambda, \Lambda', \ldots, \Lambda^\circ$ of standard tableaux of the same shape λ, such that each subsequent tableau in the chain is obtained from the previous one by an adjacent transposition of the entries. Due to the above argument, it now suffices to prove Proposition 2.3 only in the case $\Lambda = \Lambda^\circ$.

We will prove the statement by reordering the factors of the product $F_{\Lambda^\circ}(z_1, \ldots, z_n)$, using relations (7) and (8), in such a way that each singularity is part of a triple which is regular at $z_1 = z_2 = \ldots = z_n \neq 0$, and hence the whole of $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ will be manifestly regular.

Define $g(a, b)$ to be the following;

$$g(a, b) = \begin{cases} F_{b-a}(w_a, w_b) & \text{if } a < b \\ 1 & \text{if } a \geqslant b \end{cases}$$

where w_i is the substitution (10).

Now, let us divide the diagram λ into two parts, consisting of those boxes with positive contents and those with non-positive contents as in Figure 1. Consider the entries of the ith column of the hook tableau Λ° of shape λ that lie below the steps. If $u_1, u_2, \ldots, u_k$ are the entries of the ith column below the steps, we define

$$C_i = \prod_{j=1}^{n} g(u_1, j) g(u_2, j) \cdots g(u_k, j). \tag{15}$$

Now consider the entries of the ith row of Λ° that lie above the steps. If $v_1, v_2, \ldots, v_l$ are the entries of the ith row above the steps, we define

$$R_i = \prod_{j=1}^{n} g(v_1, j) g(v_2, j) \cdots g(v_l, j). \tag{16}$$

Our choice of the hook tableau was such that the following is true; if d is the number of principal hooks of λ then by relations (7) and (8) we may reorder the factors of $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ such that

$$F_{\Lambda^\circ}(z_1, \ldots, z_n) = \prod_{i=1}^{d} C_i R_i.$$

Now, each singularity (a, b) has its corresponding term $F_{b-a}(w_a, w_b)$ contain in some product C_i or R_i. This singularity term will be on the immediate left of the term $F_{b-a-1}(w_{a+1}, w_b)$. Also, this ordering has been chosen such that the product of factors to the left of any such

singularity in C_i or R_i is divisible on the right by $F_{b-a-1}(w_a, w_{b+1})$.

Therefore we can replace the pair $F_{b-a}(w_a, w_b)F_{b-a-1}(w_{a+1}, w_b)$ in C_i by the triple

$$P^-_{b-a-1}F_{b-a}(w_a, w_b)F_{b-a-1}(w_{a+1}, w_b),$$

where $P^-_{b-a-1} = \frac{-1}{q+q^{-1}}F_{b-a-1}(w_a, w_{a+1})$ is the idempotent (11). Divisibility on the right by $F_{b-a-1}(w_a, w_{a+1})$ means the addition of the idempotent has no effect on the value of the product C_i. Similarly, in the product R_i we can replace the pair by

$$P^+_{b-a-1}F_{b-a}(w_a, w_b)F_{b-a-1}(w_{a+1}, w_b).$$

By Lemma 2.2, the above triples are regular at $z_1 = z_2 = \ldots = z_n \neq 0$, and therefore, so too are the products C_i and R_i, for all $1 \leqslant i \leqslant d$. Moreover, this means $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ is regular at $z_1 = z_2 = \ldots = z_n \neq 0$. ∎

Example 2.4. As an example consider the hook tableau of the Young diagram $\lambda = (3, 3, 2)$, as shown in Example 2.1.

In the original reverse-lexicographic ordering the product $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ is written as;

$$
\begin{aligned}
F_{\Lambda^\circ}(z_1, \ldots, z_n) = \ & F_1(w_1, w_2)F_2(w_1, w_3)F_1(w_2, w_3)F_3(w_1, w_4)F_2(w_2, w_4)F_1(w_3, w_4) \\
& F_4(w_1, w_5)F_3(w_2, w_5)F_2(w_3, w_5)F_1(w_4, w_5)F_5(w_1, w_6)F_4(w_2, w_6) \\
& F_3(w_3, w_6)F_2(w_4, w_6)F_1(w_5, w_6)F_6(w_1, w_7)F_5(w_2, w_7)F_4(w_3, w_7) \\
& F_3(w_4, w_7)F_3(w_4, w_7)F_2(w_5, w_7)F_1(w_6, w_7)F_7(w_1, w_8)F_6(w_2, w_8) \\
& F_5(w_3, w_8)F_4(w_4, w_8)F_3(w_5, w_8)F_2(w_6, w_8)F_1(w_7, w_8)
\end{aligned}
$$

we may now reorder this product into the form below using relations (7) and (8) as described in the above proposition. The terms in square brackets are the singularity terms with their appropriate triple terms.

$$
\begin{aligned}
F_{\Lambda^\circ}(z_1, \ldots, z_n) = \ & F_1(w_1, w_2)F_2(w_1, w_3)F_1(w_2, w_3)F_3(w_1, w_4)F_2(w_2, w_4)F_1(w_3, w_4) \\
& F_4(w_1, w_5)F_3(w_2, w_5)F_2(w_3, w_5) \, [F_5(w_1, w_6)F_4(w_2, w_6)] \, F_3(w_3, w_6) \\
& F_6(w_1, w_7) \, [F_5(w_2, w_7)F_4(w_3, w_7)] \, F_7(w_1, w_8)F_6(w_2, w_8)F_5(w_3, w_8) \\
& \cdot F_1(w_4, w_5)F_2(w_4, w_6)F_1(w_5, w_6)F_3(w_4, w_7)F_2(w_5, w_7) \\
& [F_4(w_4, w_8)F_3(w_5, w_8)] \cdot F_1(w_6, w_7)F_2(w_6, w_8)F_1(w_7, w_8)
\end{aligned}
$$

We may now add the appropriate idempotents to these singularity-triple term pairs to form triples. Since each of these triples are regular at $z_1 = z_2 = \ldots = z_n$ then so too is the whole of $F_{\Lambda^\circ}(z_1, \ldots, z_n)$. ∎

Therefore, due to the above proposition an element $F_\Lambda \in H_n$ can now be defined as the value of $F_\Lambda(z_1, \ldots, z_n)$ at $z_1 = z_2 = \ldots = z_n \neq 0$. Note that for $n = 1$ we have $F_\Lambda = 1$. For any $n \geqslant 1$, take the expansion of the element $F_\Lambda \in H_n$ in the basis of the elements T_σ where σ is ranging over S_n.

Proposition 2.5 *The coefficient in $F_\Lambda \in H_n$ of the element T_0 is 1.*

Proof. Expand the product (6) as a sum of the elements T_σ with coefficients from the field of rational functions of $z_1, \ldots, z_n$; these functions take values in $\mathbb{C}(q)$. The decomposition in S_n with ordering of the pairs (i, j) as in (6)

$$\sigma_0 = \overrightarrow{\prod_{(i,j)}} \sigma_{j-i}$$

is reduced, hence the coefficient at $T_0 = T_{\sigma_0}$ in the expansion of (6) is 1. By the definition of F_Λ, then the coefficient of T_0 in F_Λ must be also 1 ∎

In particular this shows that $F_\Lambda \neq 0$ for any nonempty diagram λ. Let us now denote by φ the involutive antiautomorphism of the algebra H_n over the field $\mathbb{C}(q)$, defined by $\varphi(T_i) = T_i$ for every $i \in 1, \ldots, n-1$.

Proposition 2.6 *The element $F_\Lambda T_0^{-1}$ is φ-invariant.*

Proof. Any element of the algebra H_n of the form $F_i(a, b)$ is φ-invariant. Hence applying the antiautomorphism φ to an element of H_n the form (6) just reverses the ordering of the factors corresponding to the pairs (i, j). Using the relations (7) and (8), we can rewrite the reversed product as

$$\overrightarrow{\prod_{(i,j)}} F_{n-j+i}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j)$$

where the pairs (i, j) are again ordered as in (6). But due to (2) and (3), we also have the identity in the algebra H_n

$$F_{n-i}(a, b)T_0 = T_0 F_i(a, b).$$

This identity along with the equality $\varphi(T_0) = T_0$ implies that any value of the function $F_\Lambda(z_1, \ldots, z_n)T_0^{-1}$ is φ-invariant. So is the element $F_\Lambda T_0^{-1} \in H_n$ ∎

Proposition 2.7 *If $\lambda = (\alpha_1, \alpha_2, \ldots, \alpha_d | \beta_1, \beta_2, \ldots, \beta_d)$ and $\mu = (\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_d | \beta_{k+1}, \beta_{k+2}, \ldots, \beta_d)$, then $F_{\Lambda^\circ} = P \cdot F_{M^\circ}$, for some element $P \in H_n$.*

Proof. Here the shape μ is obtained by removing the first k principal hooks of λ. Let x be last entry in the kth row of the hook tableau of shape λ. By the ordering described in Proposition 2.3,

$$F_{\Lambda^\circ}(z_1, \ldots, z_n) = \prod_{i=1}^{k} C_i R_i \cdot F_{M^\circ}(z_{x+1}, \ldots, z_n),$$

where C_i and R_i are defined by (15) and (16).

Since all products C_i and R_i are regular at $z_1 = z_2 = \ldots = z_n \neq 0$, Proposition 2.3 then gives us the required statement. ∎

In any given ordering of $F_{\Lambda}(z_1, \ldots, z_n)$, we want a singularity term to be placed next to an appropriate triple term such that we may then form a regular triple. In that case we will say these two terms are 'tied'. However, proving the divisibilities described in the next two propositions require some pairs to be 'untied', in which case we must form a new ordering. This is the content of the following proofs. Some explicit examples will then given in Example 2.10 below.

Proposition 2.8 *Suppose the numbers $u < v$ stand next to each other in the same column of the hook tableau Λ° of shape λ. First, let s be the last entry in the row containing u. If $c_v < 0$ then the element $F_{\Lambda^\circ} \in H_n$ is divisible on the left by $F_u(q^{2c_u(\Lambda^\circ)}, q^{2c_v(\Lambda^\circ)}) = T_u - q$. If $c_v \geqslant 0$ then the element $F_{\Lambda^\circ} \in H_n$ is divisible on the left by the product*

$$\overset{\leftarrow}{\prod_{i=u,\ldots,s}} \left(\overset{\rightarrow}{\prod_{j=s+1,\ldots,v}} F_{i+j-s-1}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)}) \right)$$

Proof. Let λ and μ be as in Proposition 2.7 with λ a partition of n and μ a partition of m. If $F_M(z_1, \ldots, z_m)$ is divisible on the left by $F_i(w_a, w_b)$ then $F_M(z_{x+1}, \ldots, z_n)$ is divisible on the left by $F_i(w_{a+x}, w_{b+x})$. Then, by Proposition 2.6, $F_M(z_{x+1}, \ldots, z_n)$ is divisible on the right by $F_{m-i}(w_{a+x}, w_{b+x})$. Therefore, by Proposition 2.7, $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ is divisible on the right by $F_{m-i}(w_{a+x}, w_{b+x})$ and so $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ is divisible on the left by $F_{n-m+i}(w_{a+x}, w_{b+x}) = F_{i+x}(w_{a+x}, w_{b+x})$. Hence we only need to prove the statement for (u, v) such that u is in the first row or first column of Λ°.

Let r be the last entry in the first column of Λ°, s the last entry in the first row of Λ°, and t the last entry in the second column of Λ°, as shown in Figure 2.

We now continue this proof by considering three cases and showing the appropriate divisibility in each.

1	$r+1$	$r+2$	$\ldots$	u	$\ldots$	$\ldots$	s
2	$s+1$	$t+1$	$\ldots$	v	$\ldots$		
$\vdots$	$\vdots$						
$\vdots$	t						
$\vdots$							
r							

Figure 2: The first two principal hooks of the hook tableau Λ°

(i) If $c_v < 0$ (i.e. u and v are in the first column of Λ°) then $v = u + 1$ and $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ can be written as $F_{\Lambda^\circ}(z_1, \ldots, z_n) = F_u(w_u, w_v) \cdot F$.

Starting with $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ written in the ordering described in Proposition 2.3 and simply moving the term $F_{v-u}(w_u, w_v)$ to the left results in all the singularity terms in the product F remaining tied to the same triple terms as originally described in that ordering, and the index of $F_{v-u}(w_u, w_v)$ increases from $u - v$ to u. Therefore we may still form regular triples for each singularity in F, and hence F is regular at $z_1 = z_2 = \ldots = z_n \neq 0$.

So by considering this expression for $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ at $z_1 = z_2 = \ldots = z_n \neq 0$ we see that F_{Λ° will be divisible on the left by $F_u(q^{2c_u(\Lambda^\circ)}, q^{2c_v(\Lambda^\circ)}) = T_{v-u} - q$.

(ii) If $c_v = 0$ then $v = s + 1$, and $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ can be written as

$$F_{\Lambda^\circ}(z_1, \ldots, z_n) = \overset{\leftarrow}{\prod_{i=u,\ldots,s}} F_i(w_i, w_{s+1}) \cdot F'.$$

Again, starting with the ordering described in Proposition 2.3, this results in all the singularity terms in the product F' remaining tied to the same triple terms as originally described in that ordering. Hence F' is regular at $z_1 = z_2 = \ldots = z_n \neq 0$. And so F_{Λ° is divisible on the left by

$$\overset{\leftarrow}{\prod_{i=u,\ldots,s}} F_i(q^{2c_i(\Lambda^\circ)}, q^{2c_{s+1}(\Lambda^\circ)}).$$

(iii) If $c_v > 0$ (i.e. v is above the steps) then $F_{v-u}(w_u, w_v)$ is tied to the singularity $F_{v-u+1}(w_{u-1}, w_v)$ as a triple term. To show divisibility by $F_{v-u}(w_u, w_v)$ in this case we need an alternative expression for $F_{\Lambda^\circ}(z_1, \ldots, z_n)$ that is regular when $z_1 = z_2 = \ldots = z_n \neq 0$.

Define a permutation τ as follows,

$$\tau = \overrightarrow{\prod_{i=u,\ldots,s}} \left(\overleftarrow{\prod_{j=s+1,\ldots,v}} (ij) \right)$$

$$= \begin{pmatrix} 1 & 2 & \ldots & u-1 & u & u+1 & \ldots & \ldots & \ldots & \ldots & v-1 & v & v+1 & \ldots & n \\ 1 & 2 & \ldots & u-1 & s+1 & s+2 & \ldots & v-1 & v & u & u+1 & \ldots & s-1 & s & v+1 & \ldots & n \end{pmatrix}$$

From the definition of C_1 in (15) we now define $C_1' = \psi_\tau C_1$, where ψ_τ is a homomorphism such that

$$\psi_\tau F_{j-i}(w_i, w_j) = F_{j-i}(w_i, w_{\tau j}).$$

Define R_1' as,

$$R_1' = \overleftarrow{\prod_{i=r+2,\ldots,u-1}} \left(\overrightarrow{\prod_{j=s+1,\ldots,v}} F_{i+j-s-r-1}(w_i, w_j) \right) \cdot \overrightarrow{\prod_{i=s+1,\ldots,t-1}} \left(\overrightarrow{\prod_{j=i+1,\ldots,t}} f_{j-i+1}(w_i, w_j) \right)$$

$$\times \left(\overleftarrow{\prod_{j=s+1,\ldots,t}} F_{t+1-j}(w_{r+1}, w_j) \right) \left(\overrightarrow{\prod_{j=t+1,\ldots,v}} F_{j-s}(w_{r+1}, w_j) \right)$$

$$\times \overrightarrow{\prod_{i=r+1,\ldots,s-1}} \left(\overrightarrow{\prod_{j=i+1,\ldots,s}} F_{j-i+v-s}(w_i, w_j) \right) \cdot \overrightarrow{\prod_{j=v+1,\ldots,n}} \left(\overrightarrow{\prod_{i=r+1,\ldots,s}} F_{j-i}(w_i, w_j) \right).$$

Finally, define C_2' as,

$$C_2' = \overrightarrow{\prod_{j=t+1,\ldots,n}} \left(\overrightarrow{\prod_{i=s+1,\ldots,t}} F_{j-i}(w_i, w_j) \right).$$

Then,

$$F_\Lambda(z_1, \ldots, z_n) = \overleftarrow{\prod_{i=u,\ldots,s}} \left(\overrightarrow{\prod_{j=s+1,\ldots,v}} F_{i+j-s-1}(w_i, w_j) \right) \cdot C_1' R_1' C_2' R_2 \cdot \prod_{i=3}^{d} C_i R_i,$$

where d is the number of principal hooks of λ.

The product $C_1' R_1' C_2' R_2 \cdot \prod C_i R_i$ is regular at $z_1 = z_2 = \ldots z_n \neq 0$ since, as before, for any singularity (a, b) the terms $F_i(w_a, w_b) F_{i-1}(w_{a+1}, w_b)$ can be replaced by the triple $P_{i-1}^{\pm} F_i(w_a, w_b) F_{i-1}(w_{a+1}, w_b)$ for some index i – except in the expression R_1' where the terms $F_{i-1}(w_a, w_l) F_i(w_a, w_b)$ are replaced by $F_{i-1}(w_a, w_l) F_i(w_a, w_b) P_{i-1}^{+}$, where l is the entry to the immediate left of b. Note that $l = b - 1$ when $c_b(\Lambda^\circ) > 1$ and $l = s + 1$ when $c_b(\Lambda^\circ) = 1$. And so by letting $z_1 = z_2 = \ldots = z_n \neq 0$ we see that F_{Λ° is divisible on the left by

$$\overleftarrow{\prod_{i=u,\ldots,s}} \left(\overrightarrow{\prod_{j=s+1,\ldots,v}} F_{i+j-s-1}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda)}) \right).$$

∎

Proposition 2.9 *Suppose the numbers $u < v$ stand next to each other in the same row of the hook tableau Λ° of shape λ. Let r be the last entry in the column containing u. If $c_u > 0$ then the element $F_{\Lambda^\circ} \in H_n$ is divisible on the left by $F_u(q^{2c_u(\Lambda^\circ)}, q^{c_v(\Lambda^\circ)}) = T_u + q^{-1}$. If $c_u \leqslant 0$ then the element $F_{\Lambda^\circ} \in H_n$ is divisible on the left by the product*

$$\overleftarrow{\prod_{i=u,\dots,r}} \left(\overrightarrow{\prod_{j=r+1,\dots,v}} F_{i+j-r-1}\big(q^{c_i(\Lambda^\circ)}, q^{c_j(\Lambda^\circ)}\big) \right)$$

We omit the proof of this proposition as it is very similar to that of Proposition 2.8.

Lemma 2.10 *Let Λ and $\tilde{\Lambda}$ be tableaux of the same shape such that $k = \Lambda(a, b) = \Lambda(a + 1, b) - 1$ and $\tilde{k} = \tilde{\Lambda}(a, b) = \tilde{\Lambda}(a + 1, b) - 1$. Then $F_\Lambda \in H_n$ is divisible on the left by $T_k - q$ if and only if $F_{\tilde{\Lambda}} \in H_n$ is divisible on the left by $T_{\tilde{k}} - q$.*

Proof. Let σ be the permutation such that $\tilde{\Lambda} = \sigma \cdot \Lambda$. There is a decomposition $\sigma = \sigma_{i_N} \dots \sigma_{i_1}$ such that for each $M = 1, \dots, N - 1$ the tableau $\Lambda_M = \sigma_{i_M} \dots \sigma_{i_1} \cdot \Lambda$ is standard. Note that this decomposition is not necessarily reduced.

Denote F_Λ by $F_k(q^{2c_k(\Lambda)}, q^{2c_{k+1}(\Lambda)}) \cdot F$. Then, by using the relations (7) and (8) and Proposition 2.3, we have the following chain of equalities:

$$F_{\tilde{k}}\big(q^{2c_{\tilde{k}}(\tilde{\Lambda})}, q^{2c_{\tilde{k}+1}(\tilde{\Lambda})}\big) \; \cdot \; \overleftarrow{\prod_{M=1,\dots,N}} F_{n-i_M}\big(q^{2c_{i_M+1}(\Lambda_M)}, q^{2c_{i_M}(\Lambda_M)}\big) \cdot F =$$

$$\overleftarrow{\prod_{M=1,\dots,N}} F_{i_M}\big(q^{2c_{i_M}(\Lambda_M)}, q^{2c_{i_M+1}(\Lambda_M)}\big) \; \cdot \; F_k\big(q^{2c_k(\Lambda)}, q^{2c_{k+1}(\Lambda)}\big) \cdot F =$$

$$\overleftarrow{\prod_{M=1,\dots,N}} F_{i_M}\big(q^{2c_{i_M}(\Lambda_M)}, q^{2c_{i_M+1}(\Lambda_M)}\big) \; \cdot \; F_\Lambda \; =$$

$$F_{\tilde{\Lambda}} \; \cdot \; \overleftarrow{\prod_{M=1,\dots,N}} F_{n-i_M}\big(q^{2c_{i_M+1}(\Lambda_M)}, q^{2c_{i_M}(\Lambda_M)}\big)$$

Hence divisibility by $T_k - q$ for F_Λ implies its counterpart for the tableau $\tilde{\Lambda}$ and the index $\tilde{k}$, and vice versa. Here we also use the equalities

$$F_k\big(q^{2c_k(\Lambda)}, q^{2c_{k+1}(\Lambda)}\big) = T_k - q,$$

$$F_{\tilde{k}}\big(q^{2c_{\tilde{k}}(\tilde{\Lambda})}, q^{2c_{\tilde{k}+1}(\tilde{\Lambda})}\big) = T_{\tilde{k}} - q.$$

∎

Corollary 2.11 *If $k = \Lambda(a, b)$ and $k + 1 = \Lambda(a + 1, b)$ then the element $F_\Lambda \in H_n$ is divisible on the left by $T_k - q$. If $k = \Lambda(a, b)$ and $k + 1 = \Lambda(a, b + 1)$ then the element $F_\Lambda \in H_n$ is divisible on the left by $T_k + q^{-1}$.*

Proof. Due to Lemma 2.10 it suffices to prove the first part of Corollary 2.11 for only one standard tableau Λ of shape λ. Therefore, using Proposition 2.8 and taking $\tilde{\Lambda}$ to be the hook tableau Λ° of shape λ we have shown the first part of Corollary 2.11 in the case $c_v(\Lambda) < 0$.

Next let $\Lambda^\circ(a, b) = u$, $\Lambda^\circ(a + 1, b) = v$ and s be the last entry in the row containing u. Then for $c_v(\Lambda^\circ) \geqslant 0$ Proposition 2.8 showed that $F_{\Lambda^\circ} \in H_n$ is divisible on the left by

$$\overleftarrow{\prod_{i=u,\dots,s}} \left(\overrightarrow{\prod_{j=s+1,\dots,v}} F_{i+j-s-1}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)}) \right) \tag{17}$$

Put $k = u + v - s - 1$, this is the value of the index $i + j - s - 1$ in (17) when $i = u$ and $j = v$. Let Λ be the tableau such that Λ° is obtained from the tableau $\sigma_k \cdot \Lambda$ by the permutation

$$\overleftarrow{\prod_{i=u,\dots,s}} \left(\overrightarrow{\prod_{j=s+1,\dots,v}} \sigma_{i+j-s-1} \right).$$

The tableau Λ is standard. Moreover, then $\Lambda(a, b) = k$ and $\Lambda(a + 1, b) = k + 1$. Note that the rightmost factor in the product (17), corresponding to $i = u$ and $j = v$, is

$$F_{u+v-s-1}(q^{2c_u(\Lambda^\circ)}, q^{2c_v(\Lambda^\circ)}) = T_k - q.$$

Denote by F the product of all factors in (17) but the rightmost one. Further, denote by G the product obtained by replacing each factor in F

$$F_{i+j-s-1}(q^{2c_i(\Lambda^\circ)}, q^{2c_j(\Lambda^\circ)})$$

respectively by

$$F_{n-i-j+s+1}(q^{2c_j(\Lambda^\circ)}, q^{2c_i(\Lambda^\circ)}).$$

The element $F \in H_n$ is invertible, and we have

$$F \cdot F_\Lambda = F_{\Lambda^\circ} \cdot G = F \cdot (T_k - q) \cdot (C_1' R_1' C_2' R_2 \prod C_i R_i) \cdot G,$$

where the final equality is as described in Proposition 2.8. Therefore the divisibility of the element F_{Λ° on the left by the product (17) will imply the divisibility of the element F_Λ on the left by $T_k - q$.

This shows the required divisibility for the tableau $\Lambda = \sigma \sigma_k \cdot \Lambda^\circ$. Using Lemma 2.10 again concludes the proof of the first part of Corollary 2.11.

The second part of Corollary 2.11 may be shown similarly. ∎

3   Generating irreducible representations of H_n

For every standard tableau Λ of shape λ we have defined an element F_Λ of the algebra H_n. Let us now assign to Λ another element of H_n, which will be denoted by G_Λ.

Let $\rho \in S_n$ be the permutation such that $\Lambda = \rho \cdot \Lambda^\circ$, that is $\Lambda(a,b) = \rho(\Lambda^\circ(a,b))$ for all possible a and b. For any $j = 1, \ldots, n$ denote by $\mathcal{B}_j$ the subsequence of the sequence $\rho(1), \ldots, \rho(n)$ consisting of all $i < j$ such that $\rho^{-1}(i) < \rho^{-1}(j)$. Let $|\mathcal{B}_j|$ be the length of sequence $\mathcal{B}_j$.

Consider the rational function taking values in H_n, of the variables $z_1, \ldots, z_n$

$$\overrightarrow{\prod_{j=1,\ldots,n}} \left(\overrightarrow{\prod_{k=1,\ldots,|\mathcal{B}_j|}} F_{j-k}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j) \right) \quad \text{where} \quad i = \mathcal{B}_j(k).$$

Denote this rational function by $G_\Lambda(z_1, \ldots, z_n)$.

For any $j = 1, \ldots, n$ take the subsequence of the sequence $\rho(1), \ldots, \rho(n)$ consisting of all $i < j$ such that $\rho^{-1}(i) > \rho^{-1}(j)$. Denote by $\mathcal{A}_j$ the result of reversing this subsequence. Using induction on the length of the element $\rho \in S_n$, one can prove that

$$F_\Lambda(z_1, \ldots, z_n) = G_\Lambda(z_1, \ldots, z_n) \times$$

$$\overleftarrow{\prod_{j=1,\ldots,n}} \left(\overleftarrow{\prod_{k=1,\ldots,|\mathcal{A}_j|}} F_{n-j+k}(q^{2c_i(\Lambda)} z_i, q^{2c_j(\Lambda)} z_j) \right) \quad \text{where} \quad i = \mathcal{A}_j(k).$$

Hence restriction of $G_\Lambda(z_1, \ldots, z_n)$ to the subspace $\mathcal{H}_\Lambda \subset \mathbb{C}(q)^n$ is regular on the line $(z_1 = \cdots = z_n \neq 0)$ due to Proposition 2.3. The value of that restriction is our element $G_\Lambda \in H_n$ by definition.

Note that $G_{\Lambda^\circ} = F_{\Lambda^\circ}$. Denote by V_λ the left ideal in the algebra H_n generated by the element F_{Λ°. The elements $G_\Lambda \in H_n$ for all pairwise distinct standard tableaux Λ of shape λ form a basis in the vector space V_λ, [12]. Let us now consider the left ideal $V_\lambda \subset H_n$ as H_n-module. Here the algebra H_n acts via left multiplication.

The following theorems are stated without proofs. All proofs can in found in [12, Section 3].

Theorem 3.1. *The H_n-modules V_λ for different partitions λ of n are irreducible and pairwise non-equivalent.*

Furthermore we have the following proposition about the elements G_Λ;

Proposition 3.2 *The vector $G_\Lambda \in V_\lambda$ belongs to the H_k-invariant subspace in V_λ, equivalent to the H_k-module V_κ where the partition κ is the shape of the tableau obtained by removing from Λ the entries $k+1, \ldots, n$.*

The properties of the vector G_Λ given by Proposition 3.2 for $k = 1, \ldots, n-1$, determine this vector in V_λ uniquely up to a non-zero factor from $\mathbb{C}(q)$. These properties can be restated for any irreducible H_n-module V equivalent to V_λ. Explicit formulas for the action of the generators $T_1, \ldots, T_{n-1}$ of H_n on the vectors in V determined by these properties, are known; cf. [11, Theorem 6.4].

Setting $q = 1$, the algebra H_n specializes to the symmetric group ring $\mathbb{C}S_n$. The element $T_\sigma \in H_n$ then specializes to the permutation $\sigma \in S_n$ itself. The proof of Proposition 2.3 demonstrates that the coefficients in the expansion of the element $F_\Lambda \in H_n$ relative to the basis of the elements T_σ, are regular at $q = 1$ as rational functions of the parameter q. Thus the specialization of the element $F_\Lambda \in H_n$ at $q = 1$ is well defined. The same is true for the element $G_\Lambda \in H_n$. The specializations at $q = 1$ of the basis vectors $G_\Lambda \in V_\lambda$ form the *Young seminormal basis* in the corresponding irreducible representation of the group S_n. The action of the generators $\sigma_1, \ldots, \sigma_{n-1}$ of S_n on the vectors of the latter basis was first given by [17, Theorem IV]. For the interpretation of the elements F_Λ and G_Λ using representation theory of the affine Hecke algebra $\widehat{H}_n$, see [2, Section 3] and references therein.

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